

# The Stochastic Wave Equation with Fractional Noise: a random field approach

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## Abstract

We consider the linear stochastic wave equation with spatially homogeneous Gaussian noise, which is fractional in time with index  $H > 1/2$ . We show that the necessary and sufficient condition for the existence of the solution is a relaxation of the condition obtained in [10], when the noise is white in time. Under this condition, we show that the solution is  $L^2(\Omega)$ -continuous. Similar results are obtained for the heat equation. Unlike the white noise case, the necessary and sufficient condition for the existence of the solution in the case of the heat equation is *different* (and more general) than the one obtained for the wave equation.

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## 1 Introduction

The random field approach to s.p.d.e.'s initiated in [46], has become increasingly popular in the past few decades, as an alternative to the semigroup approach developed in [13], or the analytic approach of [20].

Generally speaking, a random field solution of the (non-linear) equation:

$$Lu(t, x) = \alpha(u(t, x))\dot{W}(t, x) + \beta(u(t, x)), \quad t > 0, x \in \mathbb{R}^d \quad (1)$$

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(with vanishing initial conditions) is a collection  $\{u(t, x), t \geq 0, x \in \mathbb{R}^d\}$  of square integrable random variables, which satisfy the following integral equation:

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \alpha(u(s, y)) W(ds, dy) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \beta(u(s, y)) dy ds,$$

provided that both integrals above are well-defined (the first being a stochastic integral). In this context,  $L$  is a second-order partial differential operator with constant coefficients,  $G$  is the fundamental solution of  $Lu = 0$ , and  $\dot{W}$  is a formal way of denoting the random noise perturbing the equation.

When the equation is driven by a space-time white noise (i.e. a Gaussian noise which has the covariance structure of a Brownian motion in space-time), the random field solution exists only if the spatial dimension is  $d = 1$ . In this case, the stochastic integral above is defined with respect to a martingale measure, and the solution is well-understood for most operators  $L$ , in particular for the heat and wave operators (see [46], [7] or [43]).

To obtain a random field solution in higher dimensions, one needs to consider a different type of noise, which can be either Gaussian, but with a spatially homogenous covariance structure given formally by:

$$E[\dot{W}(t, x) \dot{W}(s, y)] = \delta(t - s) f(x - y),$$

or of Poisson type. Historically, the two approaches have been initiated at about the same time (see [27], [11], [25] for the wave equation with Gaussian noise in dimension  $d = 2$ , and [9], [41] for the Poisson case).

After the ingenious extension of the martingale measure stochastic integral due to [10], it became clear that the random field approach can be pursued for the study of general s.p.d.e.'s with spatially homogenous Gaussian noise. Since this extension allows for integrands which are non-negative measures (in space), the theory developed in [10] covers instantly the case of the (non-linear) wave equation in dimensions  $d \in \{1, 2, 3\}$ , and the case of the heat equation in any dimensions  $d$ . In the non-linear case, the existence of the solution is obtained by a Picard's iteration scheme, under the usual Lipschitz assumptions on  $\alpha, \beta$ , and the following condition, linking the operator  $L$  and the spatial covariance function  $f$ :

$$\int_{\mathbb{R}^d} \int_0^t |\mathcal{F}G(u, \cdot)(\xi)|^2 du \mu(d\xi) < \infty. \quad (2)$$

(Here  $\mu$  is a non-negative tempered measure, whose Fourier transform in  $f$ .)

Moreover, (2) is the necessary any sufficient condition for the stochastic integral  $\int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) W(ds, dy)$  to be well-defined, and hence the necessary any sufficient condition for the existence of the solution in the linear case, when  $\alpha \equiv 1$  and  $\beta \equiv 0$ . Since for both heat and wave operators,

$$c_t^{(1)} \frac{1}{1 + |\xi|^2} \leq \int_0^t |\mathcal{F}G(u, \cdot)(\xi)|^2 du \leq c_t^{(2)} \frac{1}{1 + |\xi|^2}, \quad \text{for all } \xi \in \mathbb{R}^d, \quad (3)$$

for some constants  $c_t^{(1)}, c_t^{(2)} > 0$ , condition (2) is equivalent to:

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^2} \mu(d\xi) < \infty.$$

Subsequently, using the Malliavin calculus techniques, it was shown that the random variable  $u(t, x)$  has an absolutely continuous law with respect to the Lebesgue measure on  $\mathbb{R}$ , and this density is infinitely differentiable. These results are valid for the heat equation in any dimension  $d$ , and for the wave equation in dimension  $d \in \{1, 2, 3\}$  (see [36], [37], [43]), under the additional assumption (which was removed in [30]):

$$\int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi|^2} \right)^\alpha \mu(d\xi) < \infty, \quad \text{for some } \alpha \in (0, 1). \quad (4)$$

Under (4), one also obtains the Hölder continuity of the solution for the heat equation in any dimension  $d$  and the wave equation in dimensions  $d \in \{1, 2, 3\}$ . This is done using Kolmogorov's criterion and some estimates for the  $p$ -th moments of the increments of the solution (see [39], [40], [12]).

The case of the wave equation in dimension  $d \geq 4$  was solved in the recent article [8], using an extension of the integral developed in [10]. The existence of a random-field solution is obtained under condition (2). In the affine case (i.e.  $\alpha(u) = au + b, a, b \in \mathbb{R}$  and  $\beta \equiv 0$ ), and under the additional assumption (4), the solution is shown to be Hölder continuous.

In parallel with these developments, a new process began to be used intensively in stochastic analysis: the *fractional Brownian motion* (fBm) with index  $H \in (0, 1)$ , a zero-mean Gaussian process  $(B_t)_{t \geq 0}$  with covariance:

$$R_H(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

The case  $H = 1/2$  corresponds to the classical Brownian motion, while the cases  $H > 1/2$  and  $H < 1/2$  have many contrasting properties. We refer the reader to the survey article [28] and the monographs [6] and [26] for more details. Most importantly, in the case  $H > 1/2$ ,

$$R_H(t, s) = \alpha_H \int_0^t \int_0^s |u - v|^{2H-2} du dv, \quad (5)$$

where  $\alpha_H = H(2H - 1)$ . This shows that  $(B_t)_{t \geq 0}$  has a homogenous covariance structure, similar to the spatial structure of the noise  $\dot{W}$  considered above.

Returning to our discussion about s.p.d.e.'s with a Gaussian noise, it seems natural to consider equation (1), when the covariance of the noise  $\dot{W}$  is given formally by:

$$E[\dot{W}(t, x)\dot{W}(s, y)] = \alpha_H |t - s|^{2H-2} f(x - y). \quad (6)$$

However, this simple modification changes the problem drastically, since unless  $H = 1/2$ , the fBm is *not* a semimartingale, and therefore the previous method, based on martingale measure stochastic integrals, cannot be applied.

Several methods have been proposed for developing a stochastic calculus with respect to fBm: (i) the Malliavin calculus (see [14], [1], [2], [29]), which exploits the fact that the fBm is Gaussian; (ii) the method of generalized Lebesgue-Stieltjes integration (see [47]), which uses the Hölder continuity of the fBm trajectories; (iii) the rough path analysis (see [22], [23]), which uses the fact that the paths of the fBm have bounded  $p$ -variation, for  $p > 1/H$ ; (iv) the stochastic calculus via regularization based also in general on the properties of the paths of the fBm (see [16]).

These methods have been applied to s.p.d.e.'s (see [24], [31], [42], [17]), [38]), but not using the random field approach. A notable exception is the heat equation. The linear equation with noise (6) and  $H > 1/2$  was examined in [3], for particular functions  $f$  (e.g.  $f(x) = |x|^{-(d-\alpha)}$  with  $\alpha \in (0, d)$ ). We also mention the works [32] and [44] for the case of the space variable belonging to the unit circle. The quasi-linear equation (i.e.  $\alpha \equiv 0$ ) was treated in [33], and the equation with multiplicative noise (i.e.  $\alpha(u) = u, \beta \equiv 0$ ) was studied in [18]; in these two references, the covariance structure of the noise is a particular case of (6): for  $H, H_i > 1/2$

$$E[\dot{W}(t, x)\dot{W}(s, y)] = \alpha_H |t - s|^{2H-2} \prod_{i=1}^d (\alpha_{H_i} |x_i - y_i|^{2H_i-2}).$$

(This type of noise is called *fractional Brownian field*.) The heat equation with multiplicative noise (6) was studied in [4] (for particular functions  $f$  and  $H > 1/2$ ) and [19] (in the case  $H \in (0, 1)$  and  $f = \delta_0$ ). In the case when the spatial dimension is  $d = 1$ , the non-linear equation has been treated in [38] using a two-parameter Young integral based on the Hölder continuity of fBm.

To the best of our knowledge, there is no study of the wave equation driven by a noise  $\dot{W}$ , whose covariance is given by (6). The goal of the present article is to start filling this gap, by identifying the necessary and sufficient conditions for the existence of a random field solution of the linear wave equation with noise (6) and  $H > 1/2$ . We also treat the heat equation.

When  $H > 1/2$ , it turns out that under relatively mild assumptions on the fundamental solution  $G$  of the operator  $L$ , the necessary and sufficient condition for the existence of the random-field solution of the linear equation  $Lu = \dot{W}$  is:

$$\int_{\mathbb{R}^d} \int_0^t \int_0^t \mathcal{F}G(u, \cdot)(\xi) \overline{\mathcal{F}G(v, \cdot)(\xi)} |u - v|^{2H-2} dudv \mu(d\xi) < \infty, \quad (7)$$

which is more general than (2). Note that the integrand of the  $\mu(d\xi)$  integral in (7) is the  $\mathcal{H}(0, t)$ -norm of the function  $u \mapsto \mathcal{F}G(u, \cdot)(\xi)$ . Quite surprisingly, and in contrast with (3), the estimates that we obtain for this norm are *different* in the case of the wave and heat operators: in the case of the wave equation, (7) is equivalent to

$$\int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi|^2} \right)^{H+1/2} \mu(d\xi) < \infty, \quad (8)$$

whereas in the case of the heat equation, (7) is equivalent to:

$$\int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi|^2} \right)^{2H} \mu(d\xi) < \infty, \quad (9)$$

The amazing fact is that for the wave operator, these estimates can be deduced using only the estimates of the  $L^2(0, t)$ -norm (given by (3)), the trick being to pass to the spectral representation of the  $\mathcal{H}(0, t)$ -norm of  $u \mapsto \mathcal{F}G(u, \cdot)(\xi)$ . In the case of the heat operator, there is no need for this machinery, since  $u \mapsto \mathcal{F}G(u, \cdot)(\xi)$  is a non-negative function, and its  $\mathcal{H}(0, t)$ -norm can be bounded directly by the  $L^{1/H}(0, t)$ -norm, which is easily computable.

This article is organized as follows. Section 2 contains some preliminaries, and a basic result which ensures that under (7), the stochastic integral of the fundamental solution  $G$  of the wave operator is well defined. In Section 3, we show that the solution of the wave equation exists if and only if (8) holds (Theorem 3.1). Moreover, the solution is  $L^2(\Omega)$ -continuous. Similar results are obtained in Section 4 for the heat equation, using (9). Appendix A contains some useful identities, which are needed in the sequel. Appendix B gives the spectral representation of the  $\mathcal{H}(0, t)$ -norm of the function  $\sin$ .

## 2 The Basics

We denote by  $C_0^\infty(\mathbb{R}^{d+1})$  the space of infinitely differentiable functions on  $\mathbb{R}^{d+1}$  with compact support, and  $\mathcal{S}(\mathbb{R}^d)$  the Schwartz space of rapidly decreasing  $C^\infty$  functions in  $\mathbb{R}^d$ . For  $\varphi \in L^1(\mathbb{R}^d)$ , we let  $\mathcal{F}\varphi$  be the Fourier transform of  $\varphi$ :

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \varphi(x) dx.$$

We begin by introducing the framework of [10]. Let  $\mu$  be a non-negative tempered measure on  $\mathbb{R}^d$ , i.e. a non-negative measure which satisfies:

$$\int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi|^2} \right)^l \mu(d\xi) < \infty, \quad \text{for some } l > 0.$$

Since the integrand is non-increasing in  $l$ , we may assume that  $l \geq 1$  is an integer. Note that  $1 + |\xi|^2$  behaves as a constant around 0, and as  $|\xi|^2$  at  $\infty$ , and hence (10) is equivalent to:

$$\int_{|\xi| \leq 1} \mu(d\xi) < \infty, \quad \text{and} \quad \int_{|\xi| \geq 1} \frac{1}{|\xi|^{2l}} < \infty, \quad \text{for some integer } l \geq 1. \quad (10)$$

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$  be the Fourier transform of  $\mu$  in  $\mathcal{S}'(\mathbb{R}^d)$ , i.e.

$$\int_{\mathbb{R}^d} f(x) \varphi(x) dx = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi) \mu(d\xi), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Simple properties of the Fourier transform show that for any  $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x) f(x-y) \psi(y) dx dy = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi) \overline{\mathcal{F}\psi(\xi)} \mu(d\xi).$$

An approximation argument shows that the previous equality also holds for indicator functions  $\varphi = 1_A, \psi = 1_B$ , with  $A, B \in \mathcal{B}_b(\mathbb{R}^d)$ , where  $\mathcal{B}_b(\mathbb{R}^d)$  is the class of bounded Borel sets of  $\mathbb{R}^d$ :

$$\int_A \int_B f(x-y) dx dy = \int_{\mathbb{R}^d} \mathcal{F}1_A(\xi) \overline{\mathcal{F}1_B(\xi)} \mu(d\xi). \quad (11)$$

As in [3], [4], on a complete probability space  $(\Omega, \mathcal{F}, P)$ , we consider a zero-mean Gaussian process  $W = \{W_t(A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$  with covariance:

$$E(W_t(A)W_s(B)) = R_H(t, s) \int_A \int_B f(x-y) dx dy =: \langle 1_{[0,t] \times A}, 1_{[0,s] \times B} \rangle_{\mathcal{HP}}.$$

Let  $\mathcal{E}$  be the set of linear combinations of elementary functions  $1_{[0,t] \times A}$ ,  $t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)$ , and  $\mathcal{HP}$  be the Hilbert space defined as the closure of  $\mathcal{E}$  with respect to the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{HP}}$ . (Alternatively,  $\mathcal{HP}$  can be defined as the completion of  $C_0^\infty(\mathbb{R}^{d+1})$ , with respect to the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{HP}}$ .)

The map  $1_{[0,t] \times A} \mapsto W_t(A)$  is an isometry between  $\mathcal{E}$  and the Gaussian space  $H^W$  of  $W$ , which can be extended to  $\mathcal{HP}$ . We denote this extension by:

$$\varphi \mapsto W(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \varphi(t, x) W(dt, dx).$$

In the present work, we assume that  $H > 1/2$ . Hence, (5) holds. From (11) and (5), it follows that for any  $\varphi, \psi \in \mathcal{E}$ ,

$$\begin{aligned} \langle \varphi, \psi \rangle_{\mathcal{HP}} &= \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(u, x) \psi(v, y) f(x-y) |u-v|^{2H-2} dx dy du dv \\ &= \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\varphi(u, \cdot)(\xi) \overline{\mathcal{F}\psi(v, \cdot)(\xi)} |u-v|^{2H-2} \mu(d\xi) du dv. \end{aligned}$$

Moreover, we can interchange the order of the integrals  $du dv$  and  $\mu(d\xi)$ , since for indicator functions  $\varphi$  and  $\psi$ , the integrand is a product of a function of  $(u, v)$  and a function of  $\xi$ . Hence, for  $\varphi, \psi \in \mathcal{E}$ , we have:

$$\langle \varphi, \psi \rangle_{\mathcal{HP}} = \alpha_H \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \mathcal{F}\varphi(u, \cdot)(\xi) \overline{\mathcal{F}\psi(v, \cdot)(\xi)} |u-v|^{2H-2} du dv \mu(d\xi). \quad (12)$$

The space  $\mathcal{HP}$  may contain distributions, but contains the space  $|\mathcal{HP}|$  of measurable functions  $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$  such that

$$\|\varphi\|_{|\mathcal{HP}|}^2 := \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(u, x)| |\varphi(v, y)| f(x-y) |u-v|^{2H-2} dx dy du dv < \infty.$$

We recall now several facts related to the fBm (see e.g. [28]).

Let  $B = (B_t)_{t \geq 0}$  be a fBm of index  $H > 1/2$ . For a fixed  $T > 0$ , let  $\mathcal{H}(0, T)$  be the Hilbert space defined as the closure of  $\mathcal{E}(0, T)$  (the set of step functions on  $[0, T]$ ), with respect to the inner product:

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}(0,T)} = R_H(t, s).$$

One can prove that

$$R_H(t, s) = \int_0^{t \wedge s} K_H(t, r) K_H(s, r) dr,$$

where  $K_H(t, r) = c_H^* \int_r^t (u - r)^{H-3/2} u^{H-1/2} du$  and  $c_H^* = \left( \frac{\alpha_H}{\beta(H-1/2, 2-2H)} \right)^{1/2}$ . (Here  $\beta$  denotes the Beta function.) Therefore, the map  $K_H^*$  defined by:

$$(K_H^* 1_{[0,t]})(s) = K_H(t, s) 1_{[0,t]}(s)$$

is an isometry between  $\mathcal{E}(0, T)$  and  $L^2(0, T)$ . This isometry can be extended to  $\mathcal{H}(0, T)$ , and is denoted by  $\phi \mapsto B(\phi) = \int_0^T \phi(s) dB_s$ .

The transfer operator  $K_H^*$  can be expressed in terms of fractional integrals, as follows: for any  $\phi \in \mathcal{E}(0, T)$ ,

$$(K_H^* \phi)(s) = c_H^* \Gamma(H - 1/2) s^{1/2-H} I_{T-}^{H-1/2} (u^{H-1/2} \phi(u))(s),$$

where

$$I_{T-}^\alpha f(s) = \frac{1}{\Gamma(\alpha)} \int_s^T (u - s)^{\alpha-1} f(u) du$$

denotes the fractional integral of  $f \in L^1(0, T)$ , of order  $\alpha \in (0, 1)$ .

$K_H^*$  can be extended to complex-valued functions, as follows. Let  $\mathcal{E}_{\mathbb{C}}(0, T)$  be the set of all complex linear combinations of functions  $1_{[0,t]}$ ,  $t \in [0, T]$ , and  $\mathcal{H}_{\mathbb{C}}(0, T)$  be the closure of  $\mathcal{E}_{\mathbb{C}}(0, T)$  with respect to the inner product:

$$\langle \varphi, \psi \rangle_{\mathcal{H}_{\mathbb{C}}(0,T)} = \alpha_H \int_0^T \int_0^T \varphi(u) \overline{\psi(v)} |u - v|^{2H-2} du dv.$$

The operator  $K_H^*$  is an isometry which maps  $\mathcal{H}_{\mathbb{C}}(0, T)$  onto  $L_{\mathbb{C}}^2(0, T)$  (the space of functions  $\varphi : [0, T] \rightarrow \mathbb{C}$ , with  $\int_0^T |\varphi(t)|^2 dt < \infty$ ): for any  $\phi \in \mathcal{H}_{\mathbb{C}}(0, T)$ ,

$$\alpha_H \int_0^T \int_0^T \phi(u) \overline{\phi(v)} |u - v|^{2H-2} du dv = d_H \int_0^T |I_{T-}^{H-1/2} (u^{H-1/2} \phi(u))(s)|^2 \lambda_H(ds), \quad (13)$$

where  $d_H = (c_H^*)^2 \Gamma(H - 1/2)^2$  and  $\lambda_H(ds) = s^{1-2H} ds$ .

Let  $\mathcal{E}_T$  be the class of elementary functions on  $[0, T] \times \mathbb{R}^d$ . Note that for any  $\varphi \in \mathcal{E}_T$ , the function  $t \mapsto \mathcal{F}\varphi(t, \cdot)(\xi)$  belongs to  $\mathcal{H}_{\mathbb{C}}(0, T)$ , for all  $\xi \in \mathbb{R}^d$ . Using (12) and (13), we obtain that for any  $\varphi \in \mathcal{E}_T$ ,

$$\|\varphi\|_{\mathcal{HP}}^2 = d_H \int_{\mathbb{R}^d} \int_0^T |I_{T-}^{H-1/2} (u^{H-1/2} \mathcal{F}\varphi(u, \cdot)(\xi))(s)|^2 \lambda_H(ds) \mu(d\xi) =: \|\varphi\|_0^2. \quad (14)$$

We are now ready to state our result. Note that, although the conclusion of this result resembles that of Theorem 3 of [10] (for deterministic integrands), the hypothesis are different, since the proof uses techniques specific to the fBm.

**Theorem 2.1** *Let  $[0, T] \ni t \mapsto \varphi(t, \cdot) \in \mathcal{S}'(\mathbb{R}^d)$  be a deterministic function such that  $\mathcal{F}\varphi(t, \cdot)$  is a function for all  $t \in [0, T]$ . Suppose that:*

- (i) *the function  $t \mapsto \mathcal{F}\varphi(t, \cdot)(\xi)$  belongs to  $\mathcal{H}_{\mathbb{C}}(0, T)$  for all  $\xi \in \mathbb{R}^d$ ;*
- (ii) *the function  $(t, \xi) \mapsto \mathcal{F}\varphi(t, \cdot)(\xi)$  is measurable on  $(0, T) \times \mathbb{R}^d$ ;*
- (iii)  *$\int_s^T u^{H-1/2}(u-s)^{H-3/2} |\mathcal{F}\varphi(u, \cdot)(\xi)| du < \infty$  for all  $(s, \xi) \in (0, T) \times \mathbb{R}^d$  (or  $\mathcal{F}\varphi(s, \cdot)(\xi) \geq 0$  for all  $(s, \xi) \in (0, T) \times \mathbb{R}^d$ ).*

*If*

$$I_T := \alpha_H \int_{\mathbb{R}^d} \int_0^T \int_0^T \mathcal{F}\varphi(u, \cdot)(\xi) \overline{\mathcal{F}\varphi(v, \cdot)(\xi)} |u-v|^{2H-2} dudv \mu(d\xi) < \infty, \quad (15)$$

*then  $\varphi \in \mathcal{HP}$  and  $\|\varphi\|_{\mathcal{HP}}^2 = I_T$ . (By convention, we set  $\varphi(t, \cdot) = 0$  for  $t > T$ .)*

**Remark 2.2** Conditions (i)-(iii) are satisfied by the fundamental solution  $G$  of the wave (or heat) equation. In this case,  $|\mathcal{F}G(t, \cdot)(\xi)| \leq 1$  for all  $t \geq 0$ , and hence the map  $t \mapsto \mathcal{F}G(t, \cdot)(\xi)$  belongs to  $L_{\mathbb{C}}^2(0, T)$ , which is included in  $\mathcal{H}_{\mathbb{C}}(0, T)$ .

**Proof:** The argument is a modified version of the proof of Theorem 3.8 of [3]. For any  $\xi \in \mathbb{R}^d$  fixed, we apply (13) to the function  $\phi_{\xi}(t) = \mathcal{F}\varphi(t, \cdot)(\xi)$ . We get:

$$\begin{aligned} \alpha_H \int_0^T \int_0^T \mathcal{F}\varphi(u, \cdot)(\xi) \overline{\mathcal{F}\varphi(v, \cdot)(\xi)} |u-v|^{2H-2} dudv = \\ d_H \int_0^T |I_{T-}^{H-1/2}(u^{H-1/2} \mathcal{F}\varphi(u, \cdot)(\xi))(s)|^2 \lambda_H(ds). \end{aligned} \quad (16)$$

It will be shown later that:

$$(s, \xi) \mapsto a(s, \xi) := I_{T-}^{H-1/2}(u^{H-1/2} \mathcal{F}\varphi(u, \cdot)(\xi))(s) \text{ is measurable on } (0, T) \times \mathbb{R}^d. \quad (17)$$

Hence, we can integrate with respect to  $\mu(d\xi)$  in (16). Using (15), we obtain:

$$I_T = d_H \int_{\mathbb{R}^d} \int_0^T |I_{T-}^{H-1/2}(u^{H-1/2} \mathcal{F}\varphi(u, \cdot)(\xi))(s)|^2 \lambda_H(ds) \mu(d\xi) =: \|\varphi\|_0^2 < \infty. \quad (18)$$

By the definition of  $\mathcal{HP}$  and (14), it suffices to show that for any  $\varepsilon > 0$ , there exists a function  $l = l_{\varepsilon} \in \mathcal{E}_T$  such that:

$$\|\varphi - l\|_0 < \varepsilon. \quad (19)$$

Let  $\varepsilon > 0$  be arbitrary. By (17) and (18), it follows that  $a \in L^2((0, T) \times \mathbb{R}^d, \lambda_H(ds) \times \mu(d\xi))$ . Hence, there exists a simple function  $h(s, \xi)$  such that

$$\int_{\mathbb{R}^d} \int_0^T |a(s, \xi) - h(s, \xi)|^2 \lambda_H(ds) \mu(d\xi) < \varepsilon. \quad (20)$$



Without loss of generality, we assume that  $h(s, \xi) = 1_{(c, d]}(s)1_A(\xi)$ , with  $c, d \in [0, T]$ ,  $c < d$  and  $A \in \mathcal{B}_b(\mathbb{R}^d)$ . By relation (8.1) of [35], we approximate the function  $1_{(c, d]}(s)$  in  $L^2((0, T), \lambda_H(ds))$  by  $I_{T-}^{H-1/2}(u^{H-1/2}l_0(u))(s)$  with  $l_0 \in \mathcal{E}(0, T)$ , i.e.

$$\int_0^T |1_{(c, d]}(s) - I_{T-}^{H-1/2}(u^{H-1/2}l_0(u))(s)|^2 \lambda_H(ds) < \varepsilon. \quad (21)$$

By Lemma 3.7 of [3], we approximate the function  $1_A(\xi)$  in  $L^2(\mathbb{R}^d, \mu(d\xi))$  by  $\mathcal{F}l_1(\xi)$  with  $l_1 \in \mathcal{E}(\mathbb{R}^d)$ , i.e.

$$\int_{\mathbb{R}^d} |1_A(\xi) - \mathcal{F}l_1(\xi)|^2 \mu(d\xi) < \varepsilon. \quad (22)$$

We define  $l(u, x) = l_0(u)l_1(x)$ . Clearly  $l \in \mathcal{E}_T$  and  $\mathcal{F}l(u, \cdot)(\xi) = l_0(u)\mathcal{F}l_1(\xi)$ . Let

$$b(s, \xi) := I_{T-}^{H-1/2}(u^{H-1/2}\mathcal{F}l(u, \cdot)(\xi))(s) = I_{T-}^{H-1/2}(u^{H-1/2}l_0(u))(s) \cdot \mathcal{F}l_1(\xi).$$

Using (21) and (22), we obtain that:

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_0^T |h(s, \xi) - b(s, \xi)|^2 \lambda_H(ds) \mu(d\xi) \\ & \leq 2 \left\{ \int_{\mathbb{R}^d} \int_0^T |1_{(c, d]}(s) - I_{T-}^{H-1/2}(u^{H-1/2}l_0(u))(s)|^2 1_A(\xi) \lambda_H(ds) \mu(d\xi) + \right. \\ & \quad \left. \int_{\mathbb{R}^d} \int_0^T |I_{T-}^{H-1/2}(u^{H-1/2}l_0(u))(s)|^2 |1_A(\xi) - \mathcal{F}l_1(\xi)|^2 \lambda_H(ds) \mu(d\xi) \right\} \\ & \leq 2\{\varepsilon\mu(A) + \varepsilon\|l_0\|_{\mathcal{H}(0, T)}^2/d_H\} := C_1\varepsilon. \end{aligned} \quad (23)$$

From (20) and (23), it follows that

$$\|\varphi - l\|_0^2 = d_H \int_{\mathbb{R}^d} \int_0^T |a(s, \xi) - b(s, \xi)|^2 \lambda_H(ds) \mu(d\xi) < 2d_H(\varepsilon + C_1\varepsilon) := C_2\varepsilon.$$

This concludes the proof of (19).

We now return to the proof of (17), which uses assumptions (ii) and (iii). If  $\mathcal{F}\varphi(u, \cdot)(\xi) \geq 0$ , then  $(u, s, \xi) \mapsto \phi(u, s, \xi) = 1_{\{s \leq u\}} u^{H-1/2}(u-s)^{H-3/2} \mathcal{F}\varphi(u, \cdot)(\xi)$  is measurable and non-negative, and  $a(s, \xi) = \int_0^T \phi(u, s, \xi) du$  is measurable, by Fubini's theorem.

Suppose next that  $\int_s^T u^{H-1/2}(u-s)^{H-3/2} |\mathcal{F}\varphi(u, \cdot)(\xi)| du < \infty$ . If  $l(s, \xi) = 1_{(c, d]}(s)1_A(\xi)$  is an elementary function with  $c, d \in [0, T]$ ,  $A \in \mathcal{B}_b(\mathbb{R}^d)$ , then

$$a_l(s, \xi) = I_{T-}^{H-1/2}(u^{H-1/2}l(u, \xi))(s) = 1_A(\xi) \int_s^T u^{H-1/2} 1_{(c, d]}(u) (u-s)^{H-3/2} du$$

is clearly measurable. In general, since  $(u, \xi) \mapsto \mathcal{F}\varphi(u, \cdot)(\xi)$  is measurable, there exists a sequence  $(l_n)_n$  of simple functions such that  $l_n(u, \xi) \rightarrow \mathcal{F}\varphi(u, \cdot)(\xi)$  for

all  $(u, \xi)$  and  $|l_n(u, \xi)| \leq |\mathcal{F}\varphi(u, \cdot)(\xi)|$  for all  $(u, \xi), n$  (see e.g. Theorem 13.5 of [5]). By the dominated convergence theorem, for every  $(s, \xi)$

$$|a_{l_n}(s, \xi) - a(s, \xi)| \leq \int_s^T u^{H-1/2}(u-s)^{H-3/2} |l_n(s, \xi) - \mathcal{F}\varphi(u, \cdot)(\xi)| du \rightarrow 0.$$

Since  $a_{l_n}(s, \xi)$  is measurable for every  $n$ , it follows that  $a(s, \xi)$  is measurable.  $\square$

### 3 The wave equation

We consider the linear wave equation:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t, x) &= \Delta u(t, x) + \dot{W}(t, x), \quad t > 0, x \in \mathbb{R}^d \\ u(0, x) &= 0, \quad x \in \mathbb{R}^d \\ \frac{\partial u}{\partial t}(0, x) &= 0, \quad x \in \mathbb{R}^d. \end{aligned} \tag{24}$$

Let  $G_1$  be the fundamental solution of  $u_{tt} - \Delta u = 0$ . It is known that  $G_1(t, \cdot)$  is a distribution in  $\mathcal{S}'(\mathbb{R}^d)$  with rapid decrease, and

$$\mathcal{F}G_1(t, \cdot)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \tag{25}$$

for any  $\xi \in \mathbb{R}^d, t > 0, d \geq 1$  (see e.g. [45]). In particular,

$$\begin{aligned} G_1(t, x) &= \frac{1}{2} 1_{\{|x| < t\}}, \quad \text{if } d = 1 \\ G_1(t, x) &= \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x| < t\}}, \quad \text{if } d = 2 \\ G_1(t, x) &= c_d \frac{1}{t} \sigma_t, \quad \text{if } d = 3, \end{aligned}$$

where  $\sigma_t$  denotes the surface measure on the 3-dimensional sphere of radius  $t$ .

The solution of (24) is a square-integrable process  $u = \{u(t, x); t \geq 0, x \in \mathbb{R}^d\}$  defined by:

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y) W(ds, dy).$$

By definition,  $u(t, x)$  exists if and only if the stochastic integral above is well-defined, i.e.  $g_{tx} := G_1(t - \cdot, x - \cdot) \in \mathcal{HP}$ . In this case,  $E|u(t, x)|^2 = \|g_{tx}\|_{\mathcal{HP}}^2$ .

The following theorem is the main result of this article.

**Theorem 3.1** *The solution  $u = \{u(t, x); t \geq 0, x \in \mathbb{R}^d\}$  of (24) exists if and only if the measure  $\mu$  satisfies (8). In this case, for all  $p \geq 2$  and  $T > 0$*

$$\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} E|u(t, x)|^p < \infty, \tag{26}$$

*and the map  $(t, x) \mapsto u(t, x)$  is continuous from  $\mathbb{R}_+ \times \mathbb{R}^d$  into  $L^2(\Omega)$ .*

**Example 3.2** Let  $f(x) = \gamma_{\alpha,d}|x|^{-(d-\alpha)}$  be the Riesz kernel of order  $\alpha \in (0, d)$ . Then  $\mu(d\xi) = |\xi|^{-\alpha}d\xi$  and (8) is equivalent to  $\alpha > d - 2H - 1$ .

**Example 3.3** Let  $f(x) = \gamma_\alpha \int_0^\infty w^{(\alpha-d)/2-1} e^{-w} e^{-|x|^2/(4w)} dw$  be the Bessel kernel of order  $\alpha > 0$ . Then  $\mu(d\xi) = (1 + |\xi|^2)^{-\alpha/2}$  and (8) is equivalent to  $\alpha > d - 2H - 1$ .

**Example 3.4** Let  $f(x) = \prod_{i=1}^d (\alpha_{H_i} |x_i|^{2H_i-2})$  be the covariance function of a fractional Brownian field with  $H_i > 1/2$  for all  $i = 1, \dots, d$ . Then  $\mu(d\xi) = \prod_{i=1}^d (c_{H_i} |\xi_i|^{-(2H_i-1)})$  and (8) is equivalent to  $\sum_{i=1}^d (2H_i - 1) > d - 2H - 1$ . (This can be seen using the change of variables to the polar coordinates.)

**Remark 3.5** Condition (8) is equivalent to

$$\int_{|\xi| \leq 1} \mu(d\xi) < \infty \quad \text{and} \quad \int_{|\xi| \geq 1} \frac{1}{|\xi|^{2H+1}} \mu(d\xi) < \infty.$$

**Proof of Theorem 3.1:** Note that  $g_{tx} = G_1(t - \cdot, x - \cdot)$  satisfies conditions (i)-(iii) of Theorem 2.1. Hence,  $g_{tx} \in \mathcal{HP}$  (i.e. the solution  $u$  of (24) exists) if and only if  $I_t < \infty$  for all  $t > 0$ , where

$$I_t := \alpha_H \int_{\mathbb{R}^d} \int_0^t \int_0^t \mathcal{F}g_{tx}(u, \cdot)(\xi) \overline{\mathcal{F}g_{tx}(v, \cdot)(\xi)} |u - v|^{2H-2} dudv \mu(d\xi),$$

and  $E|u(t, x)|^2 = \|g_{tx}\|_{\mathcal{HP}}^2 = I_t$ . Since  $\mathcal{F}g_{tx}(u, \cdot)(\xi) = e^{-i\xi \cdot x} \overline{\mathcal{F}G_1(t - u, \cdot)(\xi)}$ ,

$$I_t = \alpha_H \int_{\mathbb{R}^d} \int_0^t \int_0^t \mathcal{F}G_1(u, \cdot)(\xi) \overline{\mathcal{F}G_1(v, \cdot)(\xi)} |u - v|^{2H-2} dudv \mu(d\xi).$$

Using (25), we obtain:

$$I_t = \alpha_H \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^t \int_0^t \sin(u|\xi|) \sin(v|\xi|) |u - v|^{2H-2} dudv.$$

We split the integral  $\mu(d\xi)$  into two parts, which correspond to the regions  $\{|\xi| \leq 1\}$  and  $\{|\xi| \geq 1\}$ . We denote the respective integrals by  $I_t^{(1)}$  and  $I_t^{(2)}$ . Since the integrand is non-negative  $I_t < \infty$  if and only if  $I_t^{(1)} < \infty$  and  $I_t^{(2)} < \infty$ .

The fact that condition (8) is sufficient for  $I_t < \infty$  follows by Proposition 3.7 below. The necessity follows by Proposition 3.8 (using Remark 3.5).

Relation (26) with  $p = 2$  follows from the estimates obtained for  $I_t = E|u(t, x)|^2$ , using Proposition 3.7. For arbitrary  $p \geq 2$ , we use the fact that  $E|u(t, x)|^p \leq C_p (E|u(t, x)|^2)^{p/2}$ , since  $u(t, x)$  is a Gaussian random variable. The  $L^2(\Omega)$ -continuity is proved in Proposition 3.10.  $\square$

We begin with an auxiliary result. To simplify the notation, we introduce the following functions: for  $\lambda > 0, \tau > 0$ , let

$$f_t(\lambda, \tau) = \sin \tau \lambda t - \tau \sin \lambda t, \quad g_t(\lambda, \tau) = \cos \tau \lambda t - \cos \lambda t. \quad (27)$$

**Lemma 3.6** For any  $\lambda > 0$  and  $t > 0$ ,

$$c_t^{(1)} \frac{\lambda^3}{1 + \lambda^2} \leq \int_{\mathbb{R}} \frac{1}{(\tau^2 - 1)^2} [f_t^2(\lambda, \tau) + g_t^2(\lambda, \tau)] d\tau \leq c_t^{(2)} \frac{\lambda^3}{1 + \lambda^2},$$

where  $c_t^{(1)} = c_1(t \wedge t^3)$  and  $c_t^{(2)} = c_2(t + t^3)$ , for some positive constants  $c_1, c_2$ .

**Proof:** From the proof of Lemma B.1, we see that:

$$\frac{1}{(\tau^2 - 1)^2} [f_t^2(\lambda, \tau) + g_t^2(\lambda, \tau)] = |\mathcal{F}_{0, \lambda t} \varphi(\tau)|^2,$$

where  $\varphi(x) = \sin x$ . Using the Plancharel's identity (39), we obtain:

$$\begin{aligned} \int_{\mathbb{R}} \frac{1}{(\tau^2 - 1)^2} [f_t^2(\lambda, \tau) + g_t^2(\lambda, \tau)] d\tau &= \int_{\mathbb{R}} |\mathcal{F}_{0, \lambda t} \varphi(\tau)|^2 d\tau = \\ &= 2\pi \int_0^{\lambda t} |\sin x|^2 dx = 2\pi \lambda \int_0^t |\sin \lambda s|^2 ds = 2\pi \lambda^3 \int_0^t \frac{|\sin \lambda s|^2}{\lambda^2} ds \end{aligned}$$

The result follows using (3): (see e.g. Lemma 6.1.2) of [43])

$$c_t^{(1)} \frac{1}{1 + \lambda^2} \leq \int_0^t \frac{|\sin \lambda s|^2}{\lambda^2} ds \leq c_t^{(2)} \frac{1}{1 + \lambda^2}.$$

□

We denote by  $N_t(\xi)$  the  $\mathcal{H}(0, t)$ -norm of  $u \mapsto \mathcal{F}G_1(u, \cdot)(\xi)$ , i.e.

$$N_t(\xi) = \frac{\alpha_H}{|\xi|^2} \int_0^t \int_0^t \sin(u|\xi|) \sin(v|\xi|) |u - v|^{2H-2} du dv.$$

**Proposition 3.7** For any  $t > 0, \xi \in \mathbb{R}^d$

$$\begin{aligned} N_t(\xi) &\leq C_H t^{2H+2} \left( \frac{1}{1 + |\xi|^2} \right)^{H+1/2}, \quad \text{if } |\xi| \leq 1 \\ N_t(\xi) &\leq c_{t,H}^{(3)} \left( \frac{1}{1 + |\xi|^2} \right)^{H+1/2}, \quad \text{if } |\xi| \geq 1 \end{aligned}$$

where  $C_H = b_H^2 2^{H+1/2}/3$  and  $c_{t,H}^{(3)} = c_H (\frac{C}{1-H} + c_t^{(2)}) 2^{3H-1/2}$ . Here  $c_t^{(2)}$  is the constant given by Lemma 3.6.

**Proof:** a) Suppose that  $|\xi| \leq 1$ . We use the fact that  $\|\varphi\|_{\mathcal{H}(0,t)}^2 \leq b_H^2 \|\varphi\|_{L^{1/H}(0,t)}^2 \leq b_H^2 t^{2H-1} \|\varphi\|_{L^2(0,t)}^2$  for any  $\varphi \in L^2(0, t)$ , and  $|\sin x| \leq x$  for any  $x > 0$ . Hence,

$$\begin{aligned} N_t(\xi) &\leq b_H^2 t^{2H-1} \frac{1}{|\xi|^2} \int_0^t \sin^2(u|\xi|) du \leq b_H^2 t^{2H-1} \int_0^t u^2 du \\ &= b_H^2 t^{2H-1} \frac{t^3}{3} \leq \frac{1}{3} b_H^2 t^{2H+2} 2^{H+1/2} \left( \frac{1}{1 + |\xi|^2} \right)^{H+1/2}, \end{aligned}$$

where for the last inequality we used the fact that  $\frac{1}{2} \leq \frac{1}{1+|\xi|^2}$  if  $|\xi| \leq 1$ .

b) Suppose that  $|\xi| \geq 1$ . Using the change of variable  $u' = u|\xi|$ ,  $v' = v|\xi|$ ,

$$\begin{aligned} N_t(\xi) &= \frac{\alpha_H}{|\xi|^{2H+2}} \int_0^{t|\xi|} \int_0^{t|\xi|} \sin(u') \sin(v') |u' - v'|^{2H-2} du' dv' \\ &= \frac{1}{|\xi|^{2H+2}} \|\sin(\cdot)\|_{\mathcal{H}(0,t|\xi|)}^2. \end{aligned}$$

Using the expression of the  $\mathcal{H}(0,t|\xi|)$ -norm of  $\sin(\cdot)$  given by Lemma B.1, we obtain:

$$N_t(\xi) = \frac{c_H}{|\xi|^{2H+2}} \int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2 - 1)^2} [f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)] d\tau. \quad (28)$$

We split the integral into the regions  $|\tau| \leq 1/2$  and  $|\tau| \geq 1/2$ , and we denote the two integrals by  $N_t^{(1)}(\xi)$  and  $N_t^{(2)}(\xi)$ .

Since  $|f_t(\lambda, \tau)| \leq 1 + |\tau|$  and  $|g_t(\lambda, \tau)| \leq 2$  for any  $\lambda > 0, \tau > 0$ , we have:

$$\begin{aligned} N_t^{(1)}(\xi) &\leq c_H \frac{1}{|\xi|^{2H+2}} \int_{|\tau| \leq 1/2} \frac{|\tau|^{-(2H-1)}}{(1 - \tau^2)^2} [(1 + |\tau|)^2 + 4] d\tau \\ &\leq c_H \frac{1}{|\xi|^{2H+1}} \int_{|\tau| \leq 1/2} C |\tau|^{-(2H-1)} d\tau \\ &= C \frac{c_H}{1-H} \left(\frac{1}{2}\right)^{2-2H} \frac{1}{|\xi|^{2H+1}}. \end{aligned}$$

We used the fact that  $|\xi|^{2H+2} \geq |\xi|^{2H+1}$  if  $|\xi| \geq 1$ , and  $\frac{1}{(1-\tau^2)^2} [(1 + |\tau|)^2 + 4] \leq \frac{1}{(3/4)^2} [(3/2)^2 + 4] = C$  if  $|\tau| \leq 1/2$ .

Using the fact that  $|\tau|^{-(2H-1)} \leq (\frac{1}{2})^{-(2H-1)}$  if  $|\tau| \geq \frac{1}{2}$ , Lemma 3.6, and the fact that  $|\xi|^2/(1 + |\xi|^2) \leq 1$ , we obtain:

$$\begin{aligned} N_t^{(2)}(\xi) &\leq \frac{c_H}{2^{-(2H-1)}} \frac{1}{|\xi|^{2H+2}} \int_{|\tau| \geq 1/2} \frac{1}{(\tau^2 - 1)^2} [f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)] d\tau \\ &\leq \frac{c_H}{2^{-(2H-1)}} \frac{1}{|\xi|^{2H+2}} \int_{\mathbb{R}} \frac{1}{(\tau^2 - 1)^2} [f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)] d\tau \\ &\leq \frac{c_H}{2^{-(2H-1)}} c_t^{(2)} \frac{1}{|\xi|^{2H+2}} \cdot |\xi| \frac{|\xi|^2}{1 + |\xi|^2} \\ &\leq \frac{c_H}{2^{-(2H-1)}} c_t^{(2)} \frac{1}{|\xi|^{2H+1}}. \end{aligned}$$

□

**Proposition 3.8** a) If  $I_t^{(1)} < \infty$  for  $t = 1$ , then  $\int_{|\xi| \leq 1} \mu(d\xi) < \infty$ .

b) Let  $l \geq 1$  be the integer from (10) and  $m = 2l - 2$ . For any  $t > 0$ ,

$$\int_{|\xi| \geq 1} \frac{\mu(d\xi)}{|\xi|^{2H+1}} \leq a_{H,t} \left( \sum_{i=0}^m b_t^i \right) I_t^{(2)} + b_t^{m+1} \int_{|\xi| \geq 1} \frac{\mu(d\xi)}{|\xi|^{2H+2+m}}, \quad (29)$$

where  $a_{H,t} = 2^{2H}/(c_H c_t^{(1)})$ ,  $b_t = 2C/c_t^{(1)}$  and  $c_t^{(1)}$  is the constant of Lemma 3.6.

In particular, if  $I_t^{(2)} < \infty$  for some  $t > 0$ , then  $\int_{|\xi| \geq 1} |\xi|^{-(2H+1)} \mu(d\xi) < \infty$ .

**Proof:** a) Using the fact that  $\sin x/x \geq \sin 1$  for all  $x \in [0, 1]$ , we have:

$$\begin{aligned} I_1^{(1)} &= \int_{|\xi| \leq 1} \frac{\mu(d\xi)}{|\xi|^2} \int_0^1 \int_0^1 \sin(u|\xi|) \sin(v|\xi|) |u-v|^{2H-2} du dv \\ &\geq \sin^2 1 \int_{|\xi| \leq 1} \mu(d\xi) \int_0^1 \int_0^1 uv |u-v|^{2H-2} du dv. \end{aligned}$$

b) According to (28),

$$I_t^{(2)} = c_H \int_{|\xi| \geq 1} \frac{\mu(d\xi)}{|\xi|^{2H+2}} \int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2 - 1)^2} [f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)] d\tau. \quad (30)$$

For any  $k \in \{-1, 0, \dots, m\}$ , let

$$I(k) := \int_{|\xi| \geq 1} \frac{1}{|\xi|^{2H+2+k}} \mu(d\xi).$$

By (10),  $I(m) = \int_{|\xi| \geq 1} |\xi|^{-(2H+2+m)} \mu(d\xi) \leq \int_{|\xi| \geq 1} |\xi|^{-2l} \mu(d\xi) < \infty$ .

We will prove that the integrals  $I(k)$  satisfy a certain recursive relation. By reverse induction, this will imply that all integrals  $I(k)$  with  $k \in \{-1, 0, \dots, m\}$  are finite. For this, for  $k \in \{0, 1, \dots, m\}$ , we let

$$A_t(k) := \int_{|\xi| \geq 1} \frac{\mu(d\xi)}{|\xi|^{2H+2+k}} \int_{\mathbb{R}} \frac{1}{(\tau^2 - 1)^2} [f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)] d\tau. \quad (31)$$

We consider separately the regions  $\{|\tau| \leq 2\}$  and  $\{|\tau| \geq 2\}$ . For the region  $\{|\tau| \leq 2\}$ , we use the expression (30) of  $I_t^{(2)}$ . Using the fact that  $|\xi|^{2H+2+k} \geq |\xi|^{2H+2}$  (since  $k \geq 0$ ), and  $|\tau|^{-(2H-1)} \geq 2^{-(2H-1)}$  if  $|\tau| \leq 2$ , we obtain:

$$\begin{aligned} A'_t(k) &:= \int_{|\xi| \geq 1} \frac{\mu(d\xi)}{|\xi|^{2H+2+k}} \int_{|\tau| \leq 2} \frac{1}{(\tau^2 - 1)^2} [f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)] d\tau \\ &\leq 2^{2H-1} \int_{|\xi| \geq 1} \frac{\mu(d\xi)}{|\xi|^{2H+2}} \int_{|\tau| \leq 2} \frac{|\tau|^{-(2H-1)}}{(\tau^2 - 1)^2} [f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)] d\tau \\ &\leq 2^{2H-1} \frac{1}{c_H} I_t^{(2)}, \quad \text{by (30)}. \end{aligned}$$

For the region  $\{|\tau| \geq 2\}$ , we use the fact  $|f_t(\lambda, \tau)| \leq 1 + |\tau|$  and  $|g_t(\lambda, \tau)| \leq 2$  for all  $\lambda > 0, \tau > 0$ . Hence,

$$\begin{aligned} A''_t(k) &:= \int_{|\xi| \geq 1} \frac{\mu(d\xi)}{|\xi|^{2H+2+k}} \int_{|\tau| \geq 2} \frac{1}{(\tau^2 - 1)^2} [f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)] d\tau \\ &\leq \int_{|\xi| \geq 1} \frac{\mu(d\xi)}{|\xi|^{2H+2+k}} \int_{|\tau| \geq 2} \frac{1}{(\tau^2 - 1)^2} [(1 + |\tau|)^2 + 4] d\tau = CI(k). \end{aligned}$$

Hence, for any  $k \in \{0, 1, \dots, m\}$

$$A_t(k) \leq 2^{2H-1} \frac{1}{c_H} I_t^{(2)} + CI(k).$$

Using Lemma 3.6, and the fact that  $\frac{|\xi|^2}{1+|\xi|^2} \geq \frac{1}{2}$  if  $|\xi| \geq 1$ , we obtain:

$$A_t(k) \geq c_t^{(1)} \int_{|\xi| \geq 1} \frac{\mu(d\xi)}{|\xi|^{2H+2+k}} \cdot \frac{|\xi|^3}{1+|\xi|^2} \geq \frac{1}{2} c_t^{(1)} I(k-1),$$

for all  $k \in \{0, 1, \dots, m\}$ . From the last two relations, we conclude that:

$$\frac{1}{2} c_t^{(1)} I(k-1) \leq 2^{2H-1} \frac{1}{c_H} I_t^{(2)} + CI(k), \quad \forall k \in \{0, 1, \dots, m\}, \quad (32)$$

or equivalently,  $I(k-1) \leq a_{H,t} I_t^{(2)} + b_t I(k)$ , for all  $k \in \{0, 1, \dots, m\}$ . Relation (29) follows by recursion.  $\square$

**Remark 3.9** In the previous argument, the recursion relation (32) uses the fact that  $k$  is *non-negative* (see the estimate of  $A'_t(k)$ ). Therefore, the “last” index  $k$  for which this relation remains true (counting downwards from  $m$ ) is  $k = 0$ , leading us to the conclusion that  $\int_{|\xi| \geq 1} |\xi|^{-(2H+1)} \mu(d\xi) < \infty$ , if  $I_t^{(2)} < \infty$ .

The next result shows that the map  $(t, x) \rightarrow u(t, x)$  from  $\mathbb{R}_+ \times \mathbb{R}^d$  into  $L^2(\Omega)$  is continuous.

**Proposition 3.10** *Suppose that (8) holds, and let  $u = \{u(t, x), t \geq 0, x \in \mathbb{R}^d\}$  be the solution of (24). For any  $t \geq 0$ ,*

$$E|u(t+h, x) - u(t, x)|^2 \rightarrow 0 \quad \text{as } |h| \rightarrow 0, \quad \text{uniformly in } x \in \mathbb{R}^d \quad (33)$$

and

$$E|u(t, x) - u(t, y)|^2 \rightarrow 0 \quad \text{as } |x - y| \rightarrow 0. \quad (34)$$

**Proof:** We use the same argument as in Lemma 19 of [10] (see also the erratum to [10]). We first show (33).

Suppose that  $h > 0$ . Splitting the interval  $[0, t+h]$  into the intervals  $[0, t]$  and  $[t, t+h]$ , and using the inequality  $|a+b|^2 \leq 2(a^2 + b^2)$ , we obtain:

$$\begin{aligned} E|u(t+h, x) - u(t, x)|^2 &\leq 2\{\|(g_{t+h,x} - g_{tx})1_{[0,t]}\|_{\mathcal{H}\mathcal{P}}^2 + \|g_{t+h,x}1_{[t,t+h]}\|_{\mathcal{H}\mathcal{P}}^2\} \\ &=: 2[E_{1,t}(h) + E_2(h)]. \end{aligned}$$

Since  $\mathcal{F}(g_{t+h,x} - g_{tx})(u, \cdot)(\xi) = e^{-i\xi \cdot x} \overline{\mathcal{F}G_1(t+h-u, \cdot)(\xi) - \mathcal{F}G_1(t-u, \cdot)(\xi)}$ ,

$$\begin{aligned} E_{1,t}(h) &= \alpha_H \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \int_0^t dv dv |u-v|^{2H-2} \mathcal{F}(g_{t+h,x} - g_{tx})(u, \cdot)(\xi) \\ &\quad \overline{\mathcal{F}(g_{t+h,x} - g_{tx})(v, \cdot)(\xi)} \\ &= \alpha_H \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \int_0^t du dv |u-v|^{2H-2} [\mathcal{F}G_1(u+h, \cdot)(\xi) - \mathcal{F}G_1(u, \cdot)(\xi)] \\ &\quad \overline{\mathcal{F}G_1(v+h, \cdot)(\xi) - \mathcal{F}G_1(v, \cdot)(\xi)} \\ &= \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} k_t(h, |\xi|), \end{aligned}$$

where

$$\begin{aligned} k_t(h, |\xi|) &= \alpha_H \int_0^t \int_0^t (\sin((u+h)|\xi|) - \sin(u|\xi|))(\sin((v+h)|\xi|) - \sin(v|\xi|)) \\ &\quad |u-v|^{2H-2} du dv = \|\sin((\cdot + h)|\xi|) - \sin(\cdot |\xi|)\|_{\mathcal{H}(0,t)}^2. \end{aligned}$$

By the Bounded Convergence Theorem,  $\lim_{h \downarrow 0} k_t(h, |\xi|) = 0$ , for any  $\xi \in \mathbb{R}^d$ .

The fact that  $E_{1,t}(h) \rightarrow 0$  as  $h \downarrow 0$  will follow from the Dominated Convergence Theorem, once we prove that:

$$k_t(h, |\xi|) \leq k_t(|\xi|), \quad \forall h \in [0, 1], \forall \xi \in \mathbb{R}^d, \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} k_t(|\xi|) < \infty. \quad (35)$$

When  $|\xi| \leq 1$ , using the same argument as in Proposition 3.7, we get:

$$\begin{aligned} k_t(h, |\xi|) &\leq b_H^2 t^{2H-1} \|\sin((\cdot + h)|\xi|) - \sin(\cdot |\xi|)\|_{L^2(0,t)}^2 \\ &\leq 2b_H^2 t^{2H-1} \left( \int_0^t \sin^2((u+h)|\xi|) du + \int_0^t \sin^2(u|\xi|) du \right) \\ &\leq 2b_H^2 t^{2H-1} |\xi|^2 \left( \int_0^t 2(u^2 + 1) du + \int_0^t u^2 du \right) =: k_t(|\xi|). \end{aligned}$$

Suppose that  $|\xi| \geq 1$ . We use the fact that:

$$k_t(h, |\xi|) \leq 2(\|\sin((\cdot + h)|\xi|)\|_{\mathcal{H}(0,t)}^2 + \|\sin(\cdot |\xi|)\|_{\mathcal{H}(0,t)}^2).$$

Using the change of variables  $u' = (u+h)|\xi|$ ,  $v' = (v+h)|\xi|$ , and (40) (Appendix A) we obtain:

$$\begin{aligned} \|\sin((\cdot + h)|\xi|)\|_{\mathcal{H}(0,t)}^2 &= \alpha_H \int_0^t \int_0^t \sin((u+h)|\xi|) \sin((v+h)|\xi|) |u-v|^{2H-2} du dv \\ &= \frac{\alpha_H}{|\xi|^{2H}} \int_{h|\xi|}^{(t+h)|\xi|} \int_{h|\xi|}^{(t+h)|\xi|} \sin(u') \sin(v') |u'-v'|^{2H-2} du' dv' \\ &= \frac{c_H}{|\xi|^{2H}} \int_{\mathbb{R}} |\mathcal{F}_{h|\xi|, (t+h)|\xi|} \varphi(\tau)|^2 |\tau|^{-(2H-1)} d\tau, \end{aligned}$$

where  $\varphi(t) = \sin t$ . Note that the square of the real part of  $\mathcal{F}_{h|\xi|, (t+h)|\xi|} \varphi(\tau)$  is:

$$\left| \int_{h|\xi|}^{(t+h)|\xi|} \cos \tau t \sin t dt \right|^2 \leq 2 \left| \int_0^{(t+h)|\xi|} \cos \tau t \sin t dt \right|^2 + 2 \left| \int_0^{h|\xi|} \cos \tau t \sin t dt \right|^2,$$

and the square of the imaginary part of  $\mathcal{F}_{h|\xi|, (t+h)|\xi|} \varphi(\tau)$  is:

$$\left| \int_{h|\xi|}^{(t+h)|\xi|} \sin \tau t \sin t dt \right|^2 \leq 2 \left| \int_0^{(t+h)|\xi|} \sin \tau t \sin t dt \right|^2 + 2 \left| \int_0^{h|\xi|} \sin \tau t \sin t dt \right|^2.$$



We now use the following fact (see Appendix B): for any  $T > 0$

$$\left| \int_0^T \cos \tau t \sin t dt \right|^2 + \left| \int_0^T \sin \tau t \sin t dt \right|^2 = \frac{1}{(\tau^2 - 1)^2} [(\sin \tau T - \tau \sin T)^2 + (\cos \tau T - \cos T)^2].$$

From here, it follows that  $k_t(h, |\xi|)$  is bounded by:

$$\frac{2c_H}{|\xi|^{2H}} \int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2 - 1)^2} [f_{t+h}^2(|\xi|, \tau) + g_{t+h}^2(|\xi|, \tau) + f_h^2(|\xi|, \tau) + g_h^2(|\xi|, \tau) + f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)] d\tau,$$

where  $f_t(\lambda, \tau)$  and  $g_t(\lambda, \tau)$  are defined by (27). The argument of Proposition 3.7 shows that for any  $t > 0$  and  $|\xi| \geq 1$

$$\int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2 - 1)^2} [f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)] d\tau \leq c_{t,H}^{(4)} \frac{|\xi|^3}{1 + |\xi|^2} \leq c_{t,H}^{(4)} |\xi|,$$

where  $c_{t,H}^{(4)} = \frac{2C}{1-H} \left(\frac{1}{2}\right)^{2-2H} + \left(\frac{1}{2}\right)^{-(2H-1)} c_t^{(2)}$ . Since  $c_{t,H}^{(4)}$  is non-decreasing in  $t$  and  $h \in [0, 1]$ ,  $k_t(h, |\xi|)$  is bounded by

$$\frac{2c_H}{|\xi|^{2H-1}} (c_{t+h,H}^{(4)} + c_{h,H}^{(4)} + c_{t,H}^{(4)}) \leq \frac{2c_H}{|\xi|^{2H-1}} (c_{t+1,H}^{(4)} + c_{1,H}^{(4)} + c_{t,H}^{(4)}) := k_t(|\xi|).$$

This concludes the proof of (35).

A similar argument shows that  $E_2(h) \rightarrow 0$  as  $h \downarrow 0$ , since

$$\begin{aligned} E_2(h) &= \alpha_H \int_{\mathbb{R}^d} \int_t^{t+h} \int_t^{t+h} \mathcal{F}G_1(t+h-u, \cdot)(\xi) \overline{\mathcal{F}G_1(t+h-v, \cdot)(\xi)} |u-v|^{2H-2} dudv \mu(d\xi) \\ &= \alpha_H \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^h \int_0^h \sin(u|\xi|) \sin(v|\xi|) |u-v|^{2H-2} dudv. \end{aligned}$$

The case  $h < 0$  is treated similarly. Using the same argument as above, it follows that for any  $h > 0$ ,  $E|u(t-h, x) - u(t, x)|^2 \leq 2(E'_{1,t}(h) + E_2(h))$ , where

$$E'_{1,t}(h) = \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} k'_t(h, |\xi|), \text{ and } k'_t(h, |\xi|) = \|\sin(\cdot |\xi|) - \sin((\cdot - h)|\xi|)\|_{\mathcal{H}(h,t)}^2.$$

To prove (34), note that

$$\begin{aligned} E|u(t, x) - u(t, y)|^2 &= \|g_{tx} - g_{ty}\|_{\mathcal{H}\mathcal{P}}^2 = \\ &= \alpha_H \int_{\mathbb{R}^d} \int_0^t \int_0^t \mathcal{F}(g_{tx} - g_{ty})(u, \cdot)(\xi) \overline{\mathcal{F}(g_{tx} - g_{ty})(v, \cdot)(\xi)} |u-v|^{2H-2} dudv \mu(d\xi) = \\ &= \alpha_H \int_{\mathbb{R}^d} |e^{-i\xi \cdot x} - e^{-i\xi \cdot y}|^2 \int_0^t \int_0^t \mathcal{F}G_1(u, \cdot)(\xi) \overline{\mathcal{F}G_1(v, \cdot)(\xi)} |u-v|^{2H-2} dudv \mu(d\xi), \end{aligned}$$

which converges to 0 as  $|x - y| \rightarrow 0$ , by the Dominated Convergence Theorem.

□

**Example 3.11** There exists an interesting connection between the solution of the wave equation with fractional noise in time and Riesz covariance in space and the odd and even parts of the fBm. Indeed, if  $f$  be the Riesz kernel of order  $\alpha \in (0, d)$ , then

$$\begin{aligned} I_t &= \alpha_H \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha-2H-2} \int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2-1)^2} [f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau)] d\tau \\ &= 2\alpha_H c_d \int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2-1)^2} \left( \int_0^\infty \frac{(\sin \tau \lambda t - \tau \sin \lambda t)^2}{\lambda^2} \lambda^{-\theta} d\lambda + \int_0^\infty \frac{(\cos \tau \lambda t - \cos \lambda t)^2}{\lambda^2} \lambda^{-\theta} d\lambda \right), \end{aligned}$$

where  $\theta = \alpha + 1 - d + 2H > 0$  under (8). If  $\theta < 1$ , the two integrals  $d\lambda$  can be expressed in terms of the covariance functions of the odd and even parts of the fBm (see [15]).

## 4 The heat equation

In this section, we consider the the heat equation with additive noise:

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) &= \frac{1}{2} \Delta u(t, x) + \dot{W}(t, x), \quad t > 0, x \in \mathbb{R}^d \\ u(0, x) &= 0, \quad x \in \mathbb{R}^d. \end{aligned} \quad (36)$$

Equation (36) was treated in [3], in the case of particular covariance kernels  $f$ . We give here an unitary approach which covers the case of any covariance kernel  $f$ , which satisfies (9).

The case of the heat equation is actually much simpler than the case of the wave equation, since both the fundamental solution  $G$  and its Fourier transform are non-negative functions.

More precisely, let  $G_2$  be the fundamental solution of  $u_t - \frac{1}{2} \Delta u = 0$ . Then

$$G_2(t, x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right), \quad t > 0, x \in \mathbb{R}^d$$

and

$$\mathcal{F}G_2(t, \cdot)(\xi) = \exp\left(-\frac{t|\xi|^2}{2}\right), \quad t > 0, \xi \in \mathbb{R}^d. \quad (37)$$

We will prove the following result.

**Theorem 4.1** *The solution  $u = \{u(t, x), t \geq 0, x \in \mathbb{R}^d\}$  of (36) exists if and only if the measure  $\mu$  satisfies (9). In this case, (26) holds for all  $p \geq 2$  and  $T > 0$ , and the solution is  $L^2(\Omega)$ -continuous.*

**Remark 4.2** (i) When  $f$  is the Riesz kernel of order  $\alpha$ , or the Bessel kernel of order  $\alpha$ , condition (9) is equivalent to  $\alpha > d - 4H$ . When  $f$  is the covariance function of the fractional Brownian field with  $H_i > 1/2$  for all  $i = 1, \dots, d$ ,

condition (9) is equivalent to  $\sum_{i=1}^d (2H_i - 1) > d - 4H$ . Note that this condition is weaker than the condition given in [33].

(ii) In Theorem 2.1 of the Erratum to [3] it has been proven that condition (9) implies that  $\|g_{tx}\|_{\mathcal{HP}} < \infty$  for any  $t \geq 0$  and  $x \in \mathbb{R}^d$ .

**Proof of Theorem 4.1:** Note that  $g_{tx} = G_2(t - \cdot, x - \cdot)$  is non-negative. Hence,  $g_{tx} \in \mathcal{HP}$  if and only if  $g_{tx} \in |\mathcal{HP}|$ . This is equivalent to saying that  $J_t := \|g_{tx}\|_{|\mathcal{HP}|}^2 < \infty$  for all  $t > 0$ . Note that

$$\begin{aligned} J_t &= \alpha_H \int_0^t \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_{tx}(u, y) g_{tx}(v, z) f(y - z) |u - v|^{2H-2} dy dz du dv \\ &= \alpha_H \int_0^t \int_0^t \int_{\mathbb{R}^d} \mathcal{F}g_{tx}(u, \cdot)(\xi) \overline{\mathcal{F}g_{tx}(v, \cdot)(\xi)} |u - v|^{2H-2} \mu(d\xi) du dv \\ &= \alpha_H \int_0^t \int_0^t \int_{\mathbb{R}^d} \mathcal{F}G_2(t - u, \cdot)(\xi) \overline{\mathcal{F}G_2(t - v, \cdot)(\xi)} |u - v|^{2H-2} \mu(d\xi) du dv. \end{aligned}$$

Using (37) and Fubini's theorem (whose application is justified since the integrand is non-negative), we obtain:

$$J_t = \alpha_H \int_{\mathbb{R}^d} \int_0^t \int_0^t \exp\left(-\frac{u|\xi|^2}{2}\right) \exp\left(-\frac{v|\xi|^2}{2}\right) |u - v|^{2H-2} du dv \mu(d\xi).$$

The existence of the solution follows from Proposition 4.3 below, which also gives estimates for  $J_t = E|u(t, x)|^2$  (and hence for  $E|u(t, x)|^p$ ). The  $L^2(\Omega)$ -continuity is given by Proposition 4.4.  $\square$

Let

$$A_t(\xi) = \alpha_H \int_0^t \int_0^t \exp\left(-\frac{u|\xi|^2}{2}\right) \exp\left(-\frac{v|\xi|^2}{2}\right) |u - v|^{2H-2} du dv$$

The next result is similar to Lemma 6.1.1) of [43].

**Proposition 4.3** *For any  $t > 0, \xi \in \mathbb{R}^d$ ,*

$$\frac{1}{4}(t^{2H} \wedge 1) \left(\frac{1}{1 + |\xi|^2}\right)^{2H} \leq A_t(\xi) \leq C_H(t^{2H} + 1) \left(\frac{1}{1 + |\xi|^2}\right)^{2H},$$

where  $C_H = b_H^2(4H)^{2H}$ .

**Proof:** Suppose that  $|\xi| \leq 1$ . Using the fact that  $\|\varphi\|_{\mathcal{H}(0,t)}^2 \leq b_H^2 t^{2H-1} \|\varphi\|_{L^2(0,t)}^2$  for all  $\varphi \in L^2(0, t)$ ,  $e^{-x} \leq 1$  for any  $x > 0$ , and  $\frac{1}{2} \leq \frac{1}{1+|\xi|^2}$  if  $|\xi| \leq 1$ ,

$$A_t(\xi) \leq b_H^2 t^{2H-1} \int_0^t \exp(-u|\xi|^2) du \leq b_H^2 t^{2H} \leq b_H^2 2^{2H} t^{2H} \left(\frac{1}{1 + |\xi|^2}\right)^{2H}.$$

Suppose that  $|\xi| \geq 1$ . Using the fact that  $\|\varphi\|_{\mathcal{H}(0,t)}^2 \leq b_H^2 \|\varphi\|_{L^{1/H}(0,t)}^2$  for any  $\varphi \in L^{1/H}(0,t)$ ,  $1 - e^{-x} \leq 1$  for all  $x > 0$ , and  $\frac{1}{|\xi|^2} \leq \frac{2}{1+|\xi|^2}$ , we obtain:

$$\begin{aligned} A_t(\xi) &\leq b_H^2 \left[ \int_0^t \exp\left(-\frac{u|\xi|^2}{2H}\right) du \right]^{2H} = b_H^2 \left( \frac{2H}{|\xi|^2} \right)^{2H} \left[ 1 - \exp\left(-\frac{t|\xi|^2}{2H}\right) \right]^{2H} \\ &\leq b_H^2 (4H)^{2H} \left( \frac{1}{1+|\xi|^2} \right)^{2H}. \end{aligned}$$

This proves the upper bound.

Next, we show the lower bound. Suppose that  $t|\xi|^2 \leq 1$ . For any  $u \in [0, t]$ ,  $\frac{u|\xi|^2}{2} \leq \frac{t|\xi|^2}{2} \leq \frac{1}{2}$ . Using the fact that  $e^{-x} \geq 1 - x$  for all  $x > 0$ , we conclude that:

$$\exp\left(-\frac{u|\xi|^2}{2}\right) \geq 1 - \frac{u|\xi|^2}{2} \geq \frac{1}{2}, \quad \forall u \in [0, t].$$

Hence

$$A_t(\xi) \geq \alpha_H \left( \frac{1}{2} \right)^2 \int_0^t \int_0^t |u - v|^{2H-2} du dv = \frac{1}{4} t^{2H} \geq \frac{1}{4} t^{2H} \left( \frac{1}{1+|\xi|^2} \right)^{2H}.$$

For the last inequality, we used the fact that  $1 \geq \frac{1}{1+|\xi|^2}$ .

Suppose that  $t|\xi|^2 \geq 1$ . Using the change of variables  $u' = u|\xi|^2/2$ ,  $v' = v|\xi|^2/2$ , we obtain:

$$A_t(\xi) = \alpha_H \frac{2^{2H}}{|\xi|^{4H}} \int_0^{t|\xi|^2/2} \int_0^{t|\xi|^2/2} e^{-u'} e^{-v'} |u' - v'|^{2H-2} du' dv'.$$

Since the integrand is non-negative,

$$\begin{aligned} A_t(\xi) &\geq \alpha_H \frac{2^{2H}}{|\xi|^{4H}} \int_0^{1/2} \int_0^{1/2} e^{-u} e^{-v} |u - v|^{2H-2} du dv \\ &= 2^{2H} \|e^{-u}\|_{\mathcal{H}(0,1/2)}^2 \frac{1}{|\xi|^{4H}} \geq 2^{2H} \left( \frac{1}{2} \right)^{2H+2} \left( \frac{1}{1+|\xi|^2} \right)^{2H}, \end{aligned}$$

where for the last inequality we used the fact that  $\frac{1}{|\xi|^2} \geq \frac{1}{1+|\xi|^2}$ , and  $\|e^{-u}\|_{\mathcal{H}(0,1/2)}^2 \geq \left( \frac{1}{2} \right)^{2H+2}$ . (This follows since  $e^{-u} \geq 1 - u \geq \frac{1}{2}$  for all  $u \in [0, \frac{1}{2}]$ .)  
□

**Proposition 4.4** *Suppose that (9) holds, and let  $u = \{u(t, x), t \geq 0, x \in \mathbb{R}^d\}$  be the mild-sense solution of (36). Then the map  $(t, x) \rightarrow u(t, x)$  from  $\mathbb{R}_+ \times \mathbb{R}^d$  into  $L^2(\Omega)$  is continuous.*

**Proof:** The argument is similar to that of Proposition 3.10. In this case, if  $h > 0$ ,

$$E_{1,t}(h) = \int_{\mathbb{R}^d} \mu(d\xi) k_t(h, |\xi|),$$

where

$$k_t(h, |\xi|) = \left\| \exp\left(-\frac{(\cdot + h)|\xi|^2}{2}\right) - \exp\left(-\frac{\cdot |\xi|^2}{2}\right) \right\|_{\mathcal{H}(0,t)}^2,$$

and

$$E_2(h) = \alpha_H \int_{\mathbb{R}^d} \mu(d\xi) \int_0^h \int_0^h \exp\left(-\frac{u|\xi|^2}{2}\right) \exp\left(-\frac{v|\xi|^2}{2}\right) |u - v|^{2H-2} du dv.$$

We omit the details.  $\square$

**Remark 4.5** We consider the operator  $Lu = \partial_t u - \sum_{i,j=1}^d a_{ij} \partial_{x_i x_j}^2 u - \sum_{i=1}^d b_i \partial_{x_i} u$ . Let  $G_3(t, x; s, y)$  be the fundamental solution of  $Lu = 0$ . We assume that:

- (i) The functions  $a_{ij}, b_i : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $i, j = 1, \dots, d$  are  $\alpha/2$ -Hölder continuous in  $t$  and  $\alpha$ -Hölder continuous in  $x$ , for some  $\alpha \in (0, 1)$ .
- (ii) There exist some  $k, K > 0$  such that for all  $(t, x) \in [0, T] \times \mathbb{R}^d, \xi \in \mathbb{R}^d$ ,

$$k|\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(t, x) \xi_i \xi_j \leq K|\xi|^2.$$

Under these assumptions,  $G_3$  is a positive function defined on  $[0, T] \times \mathbb{R}^d \times [0, T] \times \mathbb{R}^d \cap \{(s, t); 0 \leq s \leq t \leq T\}$ , which satisfies: (see p. 376 of [21])

$$G_3(t, x; s, y) \leq c_1(t-s)^{-d/2} \exp\left(-c_2 \frac{|x-y|^2}{t-s}\right) := G'_2(t-s, x-y). \quad (38)$$

Since  $G'_2(t, x)$  is essentially the same as the heat kernel  $G_2(t, x)$ , the solution of  $Lu(t, x) = \dot{W}(t, x)$  (with vanishing initial conditions) exists, if the measure  $\mu$  satisfies condition (9).

## A Some useful identities

Recall that the Fourier transform of a function  $\varphi \in L^1(\mathbb{R})$  is defined by:

$$\mathcal{F}\varphi(\tau) = \int_{\mathbb{R}} e^{-i\tau x} \varphi(x) dx.$$

For an interval  $(a, b) \subset \mathbb{R}$ , we define the restricted Fourier transform of a function  $\varphi \in L^1(a, b)$ :

$$\mathcal{F}_{a,b}\varphi(\tau) := \int_a^b e^{-i\tau x} \varphi(x) dx = \mathcal{F}(\varphi 1_{[a,b]})(\tau).$$

One can prove that  $\mathcal{F}\varphi \in L^2(\mathbb{R})$ , for any  $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . By the Plancherel's identity, for any  $\varphi, \psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ , we have:

$$\int_{\mathbb{R}} \varphi(x) \psi(x) dx = (2\pi)^{-1} \int_{\mathbb{R}} \mathcal{F}\varphi(\tau) \overline{\mathcal{F}\psi(\tau)} d\xi.$$

In particular, for any  $\varphi, \psi \in L^2(a, b)$ , we have:

$$\int_a^b \varphi(x)\psi(x)dx = (2\pi)^{-1} \int_{\mathbb{R}} \mathcal{F}_{a,b}\varphi(\tau) \overline{\mathcal{F}_{a,b}\psi(\tau)} d\tau. \quad (39)$$

(Consider  $\tilde{\varphi} = \varphi 1_{[a,b]}$ . Then  $\tilde{\varphi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  and  $\mathcal{F}\tilde{\varphi}(\xi) = \mathcal{F}_{a,b}\varphi(\xi)$ .)

The proof of Theorem 3.1 uses, in an essential way, a formula for the  $\mathcal{H}(0, T)$ -norm of  $\sin$  (developed in Appendix B), which is in turn based on the following result. (This result can be derived using for instance, the results of [34].)

**Lemma A.1** *Let  $H \in (\frac{1}{2}, 1)$ . For any  $\varphi, \psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ ,*

$$\alpha_H \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(u)\psi(v)|u-v|^{2H-2} dudv = c_H \int_{\mathbb{R}} \mathcal{F}\varphi(\tau) \overline{\mathcal{F}\psi(\tau)} |\tau|^{-(2H-1)} d\tau,$$

where  $\alpha_H = H(2H-1)$  and  $c_H = \Gamma(2H+1) \sin(\pi H)/(2\pi)$ .

*In particular, for any  $\varphi, \psi \in L^2(a, b)$ ,*

$$\alpha_H \int_a^b \int_a^b \varphi(u)\psi(v)|u-v|^{2H-2} dudv = c_H \int_{\mathbb{R}} \mathcal{F}_{a,b}\varphi(\tau) \overline{\mathcal{F}_{a,b}\psi(\tau)} |\tau|^{-(2H-1)} d\tau. \quad (40)$$

## B The $\mathcal{H}(0, T)$ -norm of $\sin$

**Lemma B.1** *Let  $\varphi(t) = \sin t$ ,  $t \in [0, T]$ . Then*

$$\|\varphi\|_{\mathcal{H}(0,T)}^2 = c_H \int_{\mathbb{R}} \frac{(\sin \tau T - \tau \sin T)^2 + (\cos \tau T - \cos T)^2}{(\tau^2 - 1)^2} |\tau|^{-(2H-1)} d\tau,$$

where  $c_H = \Gamma(2H+1) \sin(\pi H)/(2\pi)$ .

**Proof:** By (40),

$$\|\varphi\|_{\mathcal{H}(0,T)}^2 = \alpha_H \int_0^T \int_0^T \varphi(u)\varphi(v)|u-v|^{2H-2} dudv = c_H \int_{\mathbb{R}} |\mathcal{F}_{0,T}\varphi(\tau)|^2 |\tau|^{-(2H-1)} d\tau.$$

Note that  $|\mathcal{F}_{0,T}\varphi(\tau)|^2 = \left| \int_0^T e^{-i\tau t} \varphi(t) dt \right|^2 = I_1^2 + J_1^2$ , where

$$I_1 = \operatorname{Re}[\mathcal{F}_{0,T}\varphi(\tau)] = \int_0^T \cos \tau t \sin t dt, \quad J_1 = \operatorname{Im}[\mathcal{F}_{0,T}\varphi(\tau)] = \int_0^T \sin \tau t \sin t dt.$$

We calculate  $I_1$  first. Using integration by parts, we obtain:

$$I_1 = 1 - \cos \tau T \cos T - \tau I_2,$$

where  $I_2 = \int_0^T \sin \tau t \cos t dt$ . On the other hand,

$$I_1 + I_2 = \int_0^T \sin[(\tau+1)t] dt = \frac{1 - \cos[(\tau+1)T]}{\tau+1}.$$

Solving for  $I_1$  and  $I_2$ , we obtain:

$$I_1 = \frac{1}{1 - \tau^2} (1 - \cos \tau T \cos T - \tau \sin \tau T \sin T).$$

Similarly, letting  $J_2 = \int_0^T \cos \tau t \cos t dt$ , obtain:

$$\tau J_2 - J_1 = \sin \tau T \cos T \quad \text{and} \quad J_2 - J_1 = \frac{1}{\tau + 1} \sin[(\tau + 1)T].$$

Solving for  $J_1$ , we obtain:

$$J_1 = \frac{1}{1 - \tau^2} (\tau \cos \tau T \sin T - \sin \tau T \cos T).$$

An elementary calculation shows that:

$$I_1^2 + J_1^2 = \frac{1}{(1 - \tau^2)^2} [(\sin \tau T - \tau \sin T)^2 + (\cos \tau T - \cos T)^2].$$

□

**Remark B.2** Let  $B = (B_t)_{t \in \mathbb{R}}$  a fBm of index  $H$  (on the whole real line). Let  $B^o = (B_t^o)_{t \in \mathbb{R}}$  and  $B^e = (B_t^e)_{t \in \mathbb{R}}$  be the odd and even parts of  $B$  (see [15]).  $B^o$  and  $B^e$  are independent centered Gaussian processes with  $B_t = B_t^o + B_t^e$ , and

$$\begin{aligned} E(B_t^o B_s^o) &= \langle 1_{(0,t)}, 1_{(0,s)} \rangle_o := c_H \int_{\mathbb{R}} \frac{\sin \tau t \sin \tau s}{\tau^2} |\tau|^{-(2H-1)} d\tau \\ E(B_t^e B_s^e) &= \langle 1_{(0,t)}, 1_{(0,s)} \rangle_e := c_H \int_{\mathbb{R}} \frac{(1 - \cos \tau t)(1 - \cos \tau s)}{\tau^2} |\tau|^{-(2H-1)} d\tau. \end{aligned}$$

In general, for  $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ , we have:

$$\begin{aligned} E[B^o(\varphi)^2] &= \|\varphi\|_o^2 := c_H \int_{\mathbb{R}} |\operatorname{Re}[\mathcal{F}\varphi(\tau)]|^2 |\tau|^{-(2H-1)} d\tau \\ E[B^e(\varphi)^2] &= \|\varphi\|_e^2 := c_H \int_{\mathbb{R}} |\operatorname{Im}[\mathcal{F}\varphi(\tau)]|^2 |\tau|^{-(2H-1)} d\tau \end{aligned}$$

In the proof of Lemma B.1,  $I_1 = I_1(\tau)$  and  $J_1 = J_1(\tau)$  are the real and imaginary parts of  $\mathcal{F}(\varphi 1_{[0,T]})(\tau)$ , where  $\varphi(t) = \sin t$ . Hence

$$\begin{aligned} E \left( \int_0^T \sin t dB_t^o \right)^2 &= \|\sin(\cdot) 1_{[0,T]}\|_o^2 = c_H \int_{\mathbb{R}} |I_1(\tau)|^2 |\tau|^{-(2H-1)} d\tau \\ E \left( \int_0^T \sin t dB_t^e \right)^2 &= \|\sin(\cdot) 1_{[0,T]}\|_e^2 = c_H \int_{\mathbb{R}} |J_1(\tau)|^2 |\tau|^{-(2H-1)} d\tau. \end{aligned}$$

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